Mathematics - Course 221

SIMPLE APPLICATIONS OF DERIVATIVES

I Equations of Tangent and Normal to a Curve

This exercise is included to consolidate the trainee's concept of derivative as tangent slope, and to review the procedure for finding the equation of a straight line.

DEFINITION: The *normal* to the curve $y = f(x)$ at a point $P(x,y)$ on the curve is the straight line passing through P, which is perpendicular to the tangent at P.

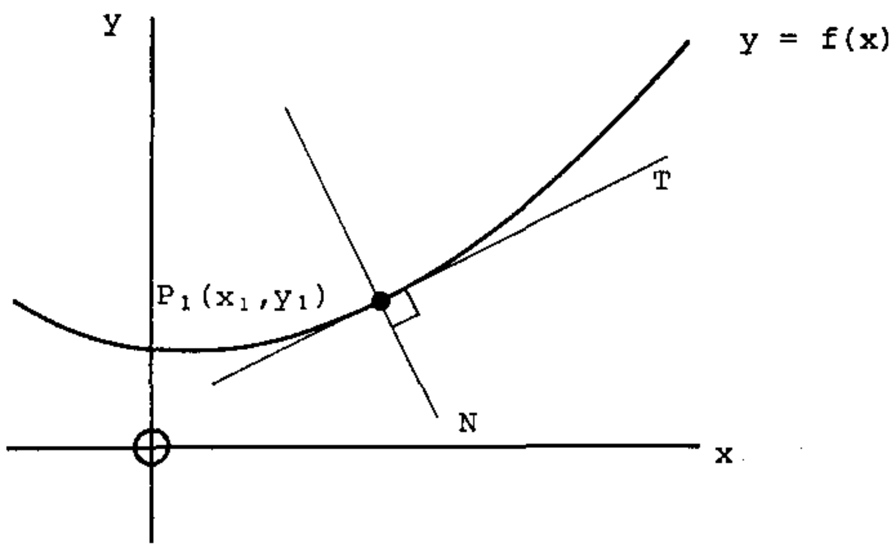


Figure 1

In Figure 1, P_1T is the tangent, and P_1N is the normal to the curve $y = f(x)$, at the point $P_1(x_1,y_1)$.

The slope of tangent $P_1T = f^1(x_1)$.

$\therefore$ Equation of tangent P_1T, is

$$\boxed{y - y_1 = f^1(x_1)(x - x_1)} \quad \text{(slope - point form)}$$

Since the slopes of perpendicular lines are negative reciprocals (cf 221.20-1),

$\therefore$ equation of normal P_1N is

$$\boxed{y - y_1 = -\frac{1}{f^1(x_1)}(x - x_1)}$$

<u>Example 1</u>

Find the equations of the tangent and normal to the curve $y = 4x - x^3$ at $x = 2$. Sketch the graph of $y = 4x - x^3$, showing tangent and normal at $x = 2$.

<u>Solution</u>

First find the y co-ordinate at x = 2, using curve equation $y = 4x - x^3$:

$$\begin{aligned} y &= 4(2) - (2)^3 \\ &= 0 \end{aligned}$$

$\therefore$ Curve, tangent and normal intersect at (2,0).

$\because \frac{dy}{dx} = 4 - 3x^2$

$\therefore$ at (2,0), tangent slope $= 4 - 3(2)^2$

$= -8$

$\therefore$ tangent equation is $y - y_1 = m(x - x_1)$

ie, $y - 0 = -8(x - 2)$

$= -8x + 16$

ie, $\underline{\underline{8x + y - 16 = 0}}$

Slope of normal $= -\frac{1}{\text{tangent slope}}$

$= -\frac{1}{-8}$

$= \frac{1}{8}$

$\therefore$ Equation of normal is $y - y_1 = m(x - x_1)$

$$\text{ie,} \quad y - 0 = \frac{1}{8}(x - 2)$$

$$\text{ie,} \quad 8y = x - 2$$

$$\text{ie,} \quad \underline{\underline{x - 8y - 2 = 0}}$$

The curve, tangent and normal are shown in Figure 2.

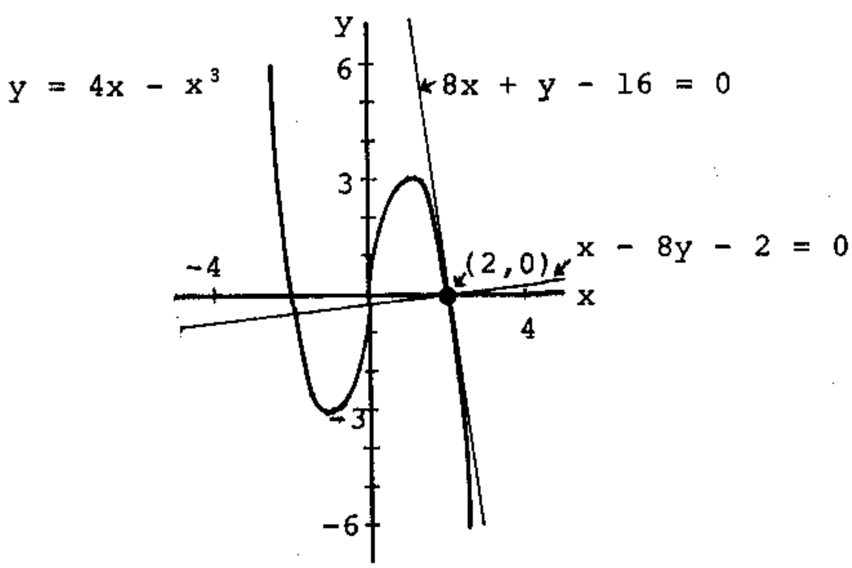


Figure 2

II Displacement, Velocity and Acceleration

The application of derivatives to such familiar concepts as velocity and acceleration should reinforce the trainee's intuitive grasp of the significance of a derivative as a rate of change.

The present discussion of displacement, velocity and acceleration will be limited to the case of motion in one dimension only.

DEFINITION: The *displacement* (designated "s") of a particle, restricted to move along an axis, is given by its co-ordinate relative to the origin on the axis.

eg, displacements of particles #1, #2, respectively, in Figure 3 are -3 and +5.

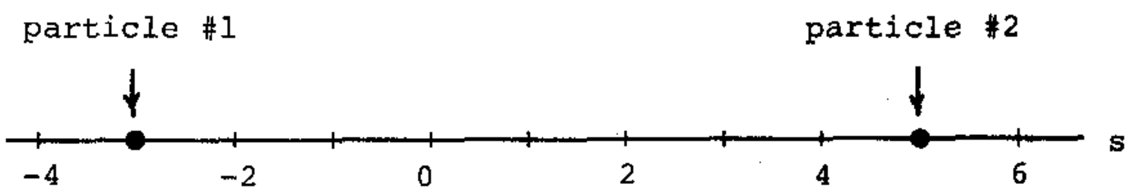


Figure 3

DEFINITION: *Velocity* (designated "v") is the rate of change of displacement with respect to time.

DEFINITION: *Acceleration* (designated "a") is the rate of change of velocity with respect to time.

Suppose a particle moving along the displacement axis passes points A and B, separated by a distance Δs, at times t_1 and $t_1 + \Delta t$, respectively (see Figure 4).

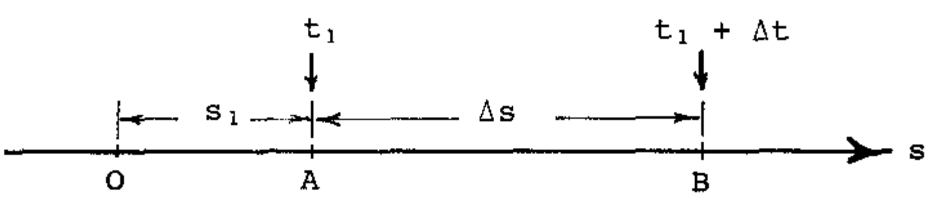


Figure 4

The particle's <u>average</u> velocity between A and B is

$$\overline{v}_{AB} = \frac{\Delta s}{\Delta t}$$

Its <u>instantaneous</u> velocity <u>AT</u> point A is

$$v_A = \lim_{B \to A} \overline{v}_{AB}$$

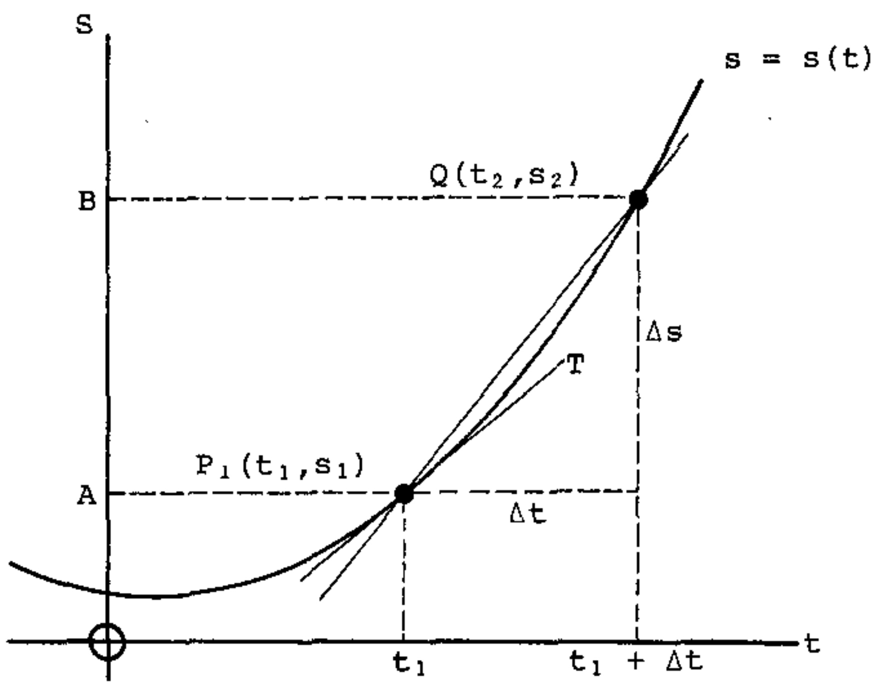
$$= \lim_{\Delta t \to o} \frac{\Delta s}{\Delta t}$$

ie, restating the above in alternative notation,

$$v(t_1) = s^1(t_1) \text{ or } \left(\frac{ds}{dt}\right)_{t = t_1},$$

where s(t) is the *displacement function.*

The connection between $\frac{ds}{dt}$ of this lesson and $\frac{dy}{dx}$ of lesson 221.20-2, will be obvious from Figure 5, which shows a typical graph of displacement versus time.



<u>Figure 5</u>

In comparing Figures 4 and 5, note that points A and B appear on the vertical axis, and instants t_1 and $t_1 + \Delta t$ on the horizontal axis of Figure 5.

The trainee should refer back to Figure 4 of lesson 221.20-2, and note its similarity to Figure 5 on previous page.

From Figure 5,

$$\underbrace{\text{instantaneous R/C "s" wrt "t"}}_{\text{instantaneous velocity, by definition}} = \lim_{Q \to P_1} (\text{slope of secant } P_1Q)$$

$$= \text{slope of tangent } P_1T$$

$$= \text{derivative of } s(t) \text{ at } t = t_1$$

Note that in this application "instantaneous" does not appear in inverted commas, because $t = t_1$ does, literally, represent an instant of time.

<u>To Summarize</u>:

$$\text{average velocity } \overline{v} = \frac{\Delta s}{\Delta t}$$

$$\text{instantaneous velocity } v(t) = \frac{ds}{dt} = \lim_{\Delta t \to o} \frac{\Delta s}{\Delta t} = s^1(t)$$

$$= \text{slope of tangent to curve } s = s(t)$$

Similar reasoning yields the following results for acceleration "a":

$$\text{average acceleration } \overline{a} = \frac{\Delta v}{\Delta t}$$

$$\text{instantaneous acceleration } a(t) = \frac{dv}{dt} = \lim_{\Delta t \to o} \frac{\Delta v}{\Delta t} = v^1(t)$$

$$= \text{slope of tangent to curve } v = v(t)$$

Example 2

Find the velocity and acceleration functions if the displacement function is

$$s(t) = 6t^2 - 4t + 2$$

Calculate the velocity and acceleration at t = 5.

Solution

Velocity function $v(t) = s^1(t)$

$$= \frac{d}{dt}(6t^2 - 4t + 2)$$

$$= \underline{\underline{12t - 4}}$$

Acceleration function $a(t) = v^1(t)$

$$= \frac{d}{dt}(12t - 4)$$

$$= \underline{\underline{12}}$$

Velocity at t = 5, $v(5) = 12(5) - 4$

$$= \underline{\underline{56}}$$

Acceleration at t = 5, $a(5) = \underline{\underline{12}}$

Example 3

If an object is thrown vertically upward with initial velocity v_0 m/s, neglecting air resistance, its displacement upwards from its starting point is given by the function

$$s(t) = v_0 t - 4.9t^2 \text{ meters.}$$

Find the time it takes a ball to reach its maximum height if thrown upward with initial velocity of 30 m/s.

Solution

$V_0 = 30 \Rightarrow s(t) = 30t - 4.9t^2$

The ball will be at maximum height when its velocity has fallen to zero. Therefore, proceed by setting the velocity equal to zero, and solving for t:

$$v(t) = s^1(t)$$
$$= 30 - 9.8t$$

$$v(t) = 0 \Rightarrow 30 - 9.8t = 0$$
$$\Rightarrow t = \frac{30}{9.8}$$
$$= 3.1$$

ie, ball reaches maximum height after <u>3.1 seconds</u>.

Example 4

Two particles have displacement functions $s_1(t) = t^3 - t$ and $s_2(t) = 6t^2 - t^3$, respectively. Find their velocities when their accelerations are equal.

Solution

Differentiate once to get the velocity functions:

$$v_1(t) = \frac{ds_1}{dt} = 3t^2 - 1, \text{ and } v_2(t) = \frac{ds_2}{dt} = 12t - 3t^2$$

Differentiate again to get the acceleration functions:

$$a_1(t) = \frac{dv_1}{dt} = 6t, \text{ and } a_2(t) = \frac{dv_2}{dt} = 12 - 6t$$

Set $a_1 = a_2$ and solve for t:

$$6t = 12 - 6t$$
$$\therefore \quad 12t = 12$$
$$\therefore \quad t = 1$$

Substitute $t = 1$ in v - functions:

$$v_1(1) = 3(1)^2 - 1$$
$$= 2$$

and $$v_2(1) = 12(1) - 3(1)^2$$
$$= 9$$

ie, particle velocities are 2 and 9 when their accelerations are equal.

ASSIGNMENT

1. Find the slope of the given curve at the given point:

   (a) $y = 8x - 3x^2$ (2,4)

   (b) $y = \frac{8}{x^2}$ (2,2)

   (c) $y = x + \frac{2}{x}$ (2,3)

2. At what point is 2 the slope of the curve $y = 4x + x^2$?

3. Find the equations of tangent and normal to the curve

   (a) $y = x\ (2 - x)^2$ at $x = 1$

   (b) $y = x^3 + 3x^{-1}$ at $x = 1$

4. Find the velocity and acceleration at $t = 2$ given the displacement function

   (a) $s(t) = 8t^2 - 3t$

   (b) $s(t) = 20 - 4t^2 - t^4$

   (c) $s(t) = \frac{10}{t}(t^3 + 8)$

5. A baseball is thrown directly upward with initial velocity 22 m/s. Neglecting air resistance, how high will it rise?

6. Given $f(x) = \frac{x^3}{3} - x^2 - 2x + 1$, find the roots of the equation $f^1(x) = 0$. What significance do these roots have for the curve $y = f(x)$? Plot $y = f(x)$. (See Appendix 3 for methods of solving quadratics).

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